Linearization of an n-link Pendulum

March 26, 2015

Assumptions and Notation:

- q_i is the angle of link i relative to a vertical line (Beware, non-standard notation: A straight joint does not need to have $q_i = 0$!)
- Center of mass m_i is always at the end of the link i. So, m_i is located at (x_i, y_i).

1 Positions, Velocities and Accelerations

The height is hence

$$\begin{aligned} y_1 &= l_1 \cos q_1 \\ y_2 &= l_1 \cos q_1 + l_2 \cos q_2 \\ &\vdots \\ y_n &= \sum\nolimits_{i=1}^{n} l_i \cos q_i \end{aligned}$$

and the horizontal displacement is

$$x_n = \sum\nolimits_{i=1}^{n} l_i \sin q_i$$

The angle around the center of mass is q_i, obviously. We have velocities

$$\begin{aligned} \dot{x}_n &= \sum\nolimits_{i=1}^{n} l_i \left(\cos q_i\right) \dot{q}_i, \\ \dot{y}_n &= -\sum\nolimits_{i=1}^{n} l_i \left(\sin q_i\right) \dot{q}_i, \end{aligned}$$

and accelerations

$$\begin{aligned} \ddot{x}_n &= \sum\nolimits_{i=1}^{n} l_i \left(\cos q_i\right) \ddot{q}_i - l_i \left(\sin q_i\right) \dot{q}_i^2, \\ \ddot{y}_n &= \sum\nolimits_{i=1}^{n} -l_i \left(\sin q_i\right) \ddot{q}_i - l_i \left(\cos q_i\right) \dot{q}_i^2. \end{aligned}$$

The ones for the rotation are obvious.

2 Forces

We compute the forces and torques

$$\begin{aligned}
\sum\nolimits_k F_{kXi} &= \left(F_{xi} - F_{x(i+1)}\right), \\
\sum\nolimits_k F_{kYi} &= \left(F_{yi} - F_{y(i+1)}\right) - m_i g, \\
\sum\nolimits_k \tau_k &= -\left(l_i \cos q_i\right) F_{xi} + \left(l_i \sin q_i\right) F_{yi} + u_i.
\end{aligned}$$

We insert into

$$\begin{aligned}
m_i \ddot{x}_i &= \sum\nolimits_k F_{kXi} = \left(F_{xi} - F_{x(i+1)}\right), \\
m_i \ddot{y}_i &= \sum\nolimits_k F_{kYi} = \left(F_{yi} - F_{y(i+1)}\right) - m_i g, \\
0 &= \sum\nolimits_k \tau_k = -\left(l_i \cos q_i\right) F_{xi} + \left(l_i \sin q_i\right) F_{yi} + u_i.
\end{aligned}$$

For $i = n$, we have $F_{x(i+1)} = F_{y(i+1)} = 0$, and so we obtain for $1 \leq i \leq n$

$$\begin{aligned}
F_{xi} &= m_i \ddot{x}_i + F_{x(i+1)} = \sum_{k=i}^{n} m_k \ddot{x}_k, \\
F_{yi} &= m_i \ddot{y}_i + F_{y(i+1)} - m_i g = \sum_{k=i}^{n} m_k \left(\ddot{y}_k - g\right), \\
0 &= -\left(l_i \cos q_i\right) \left(\sum_{k=i}^{n} m_k \ddot{x}_k\right) + \left(l_i \sin q_i\right) \left(\sum_{k=i}^{n} m_k \left(\ddot{y}_k - g\right)\right) + u_i.
\end{aligned}$$

If we insert $\ddot{x}_k$ and $\ddot{y}_k$ into the torque equation, we get

$$
\begin{aligned}
0 =& -\left(l_i \cos q_i\right) \left(\sum_{k=i}^{n} m_k \left(\sum_{j=1}^{k} l_j \left(\cos q_j\right) \ddot{q}_j - l_j \left(\sin q_j\right) \dot{q}_j^2 \right) \right) \\
& -\left(l_i \sin q_i\right) \left(\sum_{k=i}^{n} m_k \left(g + \sum_{j=1}^{k} l_j \left(\sin q_j\right) \ddot{q}_j + l_j \left(\cos q_j\right) \dot{q}_j^2 \right) \right) + u_i \\
=& u_i - \left(l_i \sin q_i\right) g \left(\sum_{k=i}^{n} m_k \right) \\
& - \sum_{k=i}^{n} m_k \sum_{j=1}^{k} l_i l_j \Bigg(\underbrace{[(\cos q_i)(\cos q_j) + (\sin q_i)(\sin q_j)]}_{\cos(q_i - q_j)} \ddot{q}_j \\
& \qquad + \underbrace{[(\sin q_i)(\cos q_j) - (\cos q_i)(\sin q_j)]}_{\sin(q_i - q_j)} \dot{q}_j^2 \Bigg) \\
=& u_i - \left(l_i \sin q_i\right) g \left(\sum_{k=i}^{n} m_k \right) - \sum_{k=i}^{n} m_k \sum_{j=1}^{k} l_i l_j \cos(q_i - q_j) \ddot{q}_j \\
& - \sum_{k=i}^{n} m_k \sum_{j=1}^{k} l_i l_j \sin(q_i - q_j) \dot{q}_j^2,
\end{aligned}
$$

As we have defined our angles absolute, we have no Coriolis forces. We can write the equations of motion by

$$
\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q})\dot{\mathbf{q}}^2 + \mathbf{g}(\mathbf{q}) = \mathbf{u}
$$

where $\dot{\mathbf{q}}^2$ is a element-wise squared $\dot{\mathbf{q}}$. We obtain

$$
\begin{aligned}
\mathbf{C}_{ih}(\mathbf{q}) &= \sum_{k=i}^{n} m_k \sum_{j=1}^{k} l_j l_i \delta_{j=h} \sin\left(q_i - q_j\right) \\
&= \sum_{k=i}^{n} m_k l_h l_i \delta_{h \le k} \sin\left(q_i - q_h\right) = l_h l_i \sin\left(q_i - q_h\right) \sum_{k=i}^{n} m_k \delta_{h \le k} \\
&= l_h l_i \sin\left(q_i - q_h\right) \sum_{k=\max(i,h)}^{n} m_k,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M}_{ih}(\mathbf{q}) &= \sum_{k=i}^{n} m_k \sum_{j=1}^{k} \delta_{j=h} l_j l_i \cos(q_i - q_j) \\
&= \sum_{k=i}^{n} (m_k l_j l_i)\, \delta_{h \le k} \cos(q_i - q_j) \\
&= l_h l_i \cos(q_i - q_h) \sum_{k=i}^{n} m_k \delta_{h \le k} \\
&= l_h l_i \cos(q_i - q_h) \sum_{k=\max(i,h)}^{n} m_k \\
\mathbf{g}_i(\mathbf{q}) &= g l_i \sin(q_i) \sum_{k=i}^{n} m_k.
\end{aligned}
$$

We need these for the next step!

3 Linearization without Analytical Matrix Inversion

Clearly a linearization of simulator

$$\ddot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q})(\mathbf{u} - \mathbf{C}(\mathbf{q})\dot{\mathbf{q}}^2 - \mathbf{g}(\mathbf{q}))$$

would be very cumbersome in terms of derivatives. However, we know that the energies can be approximated by second order Taylor expansion

$$
\begin{aligned}
T(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} \approx \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}_0)\dot{\mathbf{q}}, \\
U(\mathbf{q}) &= U(\mathbf{q}_0) + U'(\mathbf{q}_0)(\mathbf{q} - \mathbf{q}_0) + \frac{1}{2}(\mathbf{q} - \mathbf{q}_0)U''(\mathbf{q}_0)(\mathbf{q} - \mathbf{q}_0).
\end{aligned}
$$

We can get these as well by

$$
\begin{aligned}
T(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2}\sum m_i \left(\dot{x}_i^2 + \dot{y}_i^2\right) + m_i l_i^2 \dot{q}_i^2, \\
U(\mathbf{q}) &= g\sum_{i=1}^{n} m_i y_i = g\sum_{i=1}^{n} m_i \sum_{k=1}^{i} l_k \cos q_k,
\end{aligned}
$$

which in this case will be easier to work with. Note that

$$U'(\mathbf{q}_0) = -\mathbf{g}(\mathbf{q_0}),$$

and that $U''(\mathbf{q}_0)$ is a stiffness matrix. Hence, we can obtain the linearized system by

$$
\begin{aligned}
\mathbf{u} &= \frac{d}{dt}\frac{\partial (T - U)}{\partial \dot{\mathbf{q}}} - \frac{\partial (T - U)}{\partial \mathbf{q}}, \\
&= \mathbf{M}(\mathbf{q}_0)\ddot{\mathbf{q}} + U'(\mathbf{q}_0) + U''(\mathbf{q}_0)(\mathbf{q} - \mathbf{q}_0) \\
&= \mathbf{M}(\mathbf{q}_0)\ddot{\mathbf{q}} - \mathbf{g}(\mathbf{q_0}) + U''(\mathbf{q}_0)(\mathbf{q} - \mathbf{q}_0).
\end{aligned}
$$

We realize that $U''(\mathbf{q}_0)$ is a diagonal matrix with

$$U''_{ii}(\mathbf{q}) = -gl_i \cos q_i \sum_{k=i}^{n} m_k,$$

and, hence, we have a linearized simulator. We can formulate this system as a differential equation of degree 1 with

$$\begin{aligned} \mathbf{p} &= \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} = \mathbf{M_0}^{-1}\mathbf{g}(\mathbf{q_0}) + \mathbf{M_0}^{-1}U''_0\mathbf{q}_0 - \mathbf{M_0}U''_0\mathbf{q} + \mathbf{M_0}^{-1}\mathbf{u} &= \dot{\mathbf{p}}, \end{aligned}$$

where $\mathbf{M_0} = \mathbf{M}(\mathbf{q}_0)$ and $U''_0 = U''(\mathbf{q_0})$. As we have a linear system is holds for piecewise constant actions u that

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{p}} \end{pmatrix} = \frac{1}{\Delta t} \begin{pmatrix} \mathbf{q_{t+1}} - \mathbf{q_t} \\ \mathbf{p_{t+1}} - \mathbf{p_t} \end{pmatrix}$$

and we can write

$$\begin{aligned} \begin{pmatrix} \mathbf{q}_{t+1} \\ \mathbf{p}_{t+1} \end{pmatrix} = \begin{pmatrix} I & \Delta t \cdot I \\ -\Delta t \cdot \mathbf{M_0}^{-1}U''_0 & I \end{pmatrix} \begin{pmatrix} \mathbf{q}_t \\ \mathbf{p}_t \end{pmatrix} + \Delta t \cdot \begin{pmatrix} 0 \\ \mathbf{M_0}^{-1} \end{pmatrix} \mathbf{u} \\ + \Delta t \cdot \begin{pmatrix} 0 \\ \mathbf{M_0}^{-1}\mathbf{g}(\mathbf{q_0}) + \mathbf{M_0}^{-1}U''_0\mathbf{q}_0 \end{pmatrix} \end{aligned} \qquad \text{(Simulator Equation)}$$

4 Linearization around the Balance Point

The balance point, i.e. all joints are directly straight and vertically aligned, is described by $\mathbf{q}_0 = 0$. We see that $\mathbf{g}(\mathbf{q}_0) = 0$ and so the constant term in the Simulator Equation vanishes. The computation of the Mass matrix simplifies to

$$\mathbf{M}_{ih}(\mathbf{q}_0) = l_h l_i \sum_{k=\max(i,h)}^{n} m_k,$$

and

$$U''_{ii}(\mathbf{q}) = -gl_i \sum_{k=i}^{n} m_k.$$