Linearization of an $n$-link Pendulum

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Assumptions and Notation:

- $q_i$ is the angle of link $i$ relative to a vertical line (Beware, non-standard notation: A straight joint does not need to have $q_i = 0$ !)
- Center of mass $m_i$ is always at the end of the link $i$. So, $m_i$ is located at $(x_i, y_i)$.

1 Positions, Velocities and Accelerations

The height is hence

\begin{align*}
y_1 &= l_1 \cos q_1, \\
y_2 &= l_1 \cos q_1 + l_2 \cos q_2, \\
&\vdots \\
y_n &= \sum_{i=1}^{n} l_i \cos q_i
\end{align*}

and the horizontal displacement is

\begin{align*}
x_n &= \sum_{i=1}^{n} l_i \sin q_i
\end{align*}

The angle around the center of mass is $q_i$, obviously. We have velocities

\begin{align*}
\dot{x}_n &= \sum_{i=1}^{n} l_i (\cos q_i) \dot{q}_i, \\
\dot{y}_n &= -\sum_{i=1}^{n} l_i (\sin q_i) \dot{q}_i, 
\end{align*}

and accelerations

\begin{align*}
\ddot{x}_n &= \sum_{i=1}^{n} l_i (\cos q_i) \ddot{q}_i - l_i (\sin q_i) \dot{q}_i^2, \\
\ddot{y}_n &= \sum_{i=1}^{n} -l_i (\sin q_i) \ddot{q}_i - l_i (\cos q_i) \dot{q}_i^2.
\end{align*}

The ones for the rotation are obvious.
2 Forces

We compute the forces and torques

\[
\sum_k F_{kX_i} = (F_{x_i} - F_{x(i+1)}) ,
\]
\[
\sum_k F_{kY_i} = (F_{y_i} - F_{y(i+1)}) - m_i g ,
\]
\[
\sum_k \tau_k = -(l_i \cos q_i) F_{x_i} + (l_i \sin q_i) F_{y_i} + u_i .
\]

We insert into

\[
m_i \ddot{x}_i = \sum_k F_{kX_i} = (F_{x_i} - F_{x(i+1)}) ,
\]
\[
m_i \ddot{y}_i = \sum_k F_{kY_i} = (F_{y_i} - F_{y(i+1)}) - m_i g ,
\]
\[
0 = \sum_k \tau_k = -(l_i \cos q_i) F_{x_i} + (l_i \sin q_i) F_{y_i} + u_i .
\]

For \( i = n \), we have \( F_{x(i+1)} = F_{y(i+1)} = 0 \), and so we obtain for \( 1 \leq i \leq n \)

\[
F_{x_i} = m_i \ddot{x}_i + F_{x(i+1)} = \sum_{k=i}^n m_k \ddot{x}_k ,
\]
\[
F_{y_i} = m_i \ddot{y}_i + F_{y(i+1)} - m_i g = \sum_{k=i}^n m_k (\ddot{y}_k - g) ,
\]
\[
0 = -(l_i \cos q_i) \left( \sum_{k=i}^n m_k \ddot{x}_k \right) + (l_i \sin q_i) \left( \sum_{k=i}^n m_k (\ddot{y}_k - g) \right) + u_i .
\]
If we insert $\ddot{x}_k$ and $\ddot{y}_k$ into the torque equation, we get

$$0 = - (l_i \cos q_i) \left( \sum_{k=i}^n m_k \left( \sum_{j=1}^k l_j (\cos q_j) \ddot{q}_j - l_j (\sin q_j) \dot{q}_j^2 \right) \right)$$

$$- (l_i \sin q_i) \left( \sum_{k=i}^n m_k \left( g + \sum_{j=1}^k l_j (\sin q_j) \ddot{q}_j + l_j (\cos q_j) \dot{q}_j^2 \right) \right) + u_i$$

$$= u_i - (l_i \sin q_i) g \left( \sum_{k=i}^n m_k \right)$$

$$- \sum_{k=i}^n m_k \sum_{j=1}^k l_j \left( \cos(q_i-q_j) \left( \cos(q_i) + \sin(q_i) \sin(q_j) \right) \ddot{q}_j \right.$$

$$+ \left. (\sin(q_i) - \cos(q_i) \sin(q_j)) \dot{q}_j^2 \right)$$

$$= u_i - (l_i \sin q_i) g \left( \sum_{k=i}^n m_k \right) - \sum_{k=i}^n m_k \sum_{j=1}^k l_j \sin(q_i-q_j) \dot{q}_j$$

$$- \sum_{k=i}^n m_k \sum_{j=1}^k l_j \sin(q_i-q_j) \dot{q}_j^2,$$

As we have defined our angles absolute, we have no Coriolis forces. We can write the equations of motion by

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q})\dot{\mathbf{q}}^2 + \mathbf{g}(\mathbf{q}) = \mathbf{u}$$

where $\dot{\mathbf{q}}^2$ is a element-wise squared $\dot{\mathbf{q}}$. We obtain

$$\mathbf{C}_{th}(\mathbf{q}) = \sum_{k=i}^n m_k \sum_{j=1}^k l_j l_i \delta_{j=h} \sin(q_i - q_j)$$

$$= \sum_{k=i}^n m_k l_h l_i \delta_{h \leq k} \sin(q_i - q_h) = l_h l_i \sin(q_i - q_h) \sum_{k=i}^n m_k \delta_{h \leq k}$$

$$= l_h l_i \sin(q_i - q_h) \sum_{k=\max(i,h)}^n m_k,$$
\[
M_{ih}(\mathbf{q}) = \sum_{k=i}^{n} \sum_{j=1}^{k} m_k \delta_{j=h} l_j l_i \cos(q_i - q_j)
\]

\[
= \sum_{k=i}^{n} (m_k l_j l_i) \delta_{h \leq k} \cos(q_i - q_j)
\]

\[
= l_h l_i \cos(q_i - q_h) \sum_{k=i}^{n} m_k \delta_{h \leq k}
\]

\[
= l_h l_i \cos(q_i - q_h) \sum_{k=\max(i,h)}^{n} m_k
\]

\[
g_i(\mathbf{q}) = g_l \sin(q_i) \sum_{k=i}^{n} m_k.
\]

We need these for the next step!

3 Linearization without Analytical Matrix Inversion

Clearly a linearization of simulator

\[
\ddot{\mathbf{q}} = M^{-1}(\mathbf{q}) (\mathbf{u} - C(\mathbf{q}) \dot{\mathbf{q}}^2 - g(\mathbf{q}))
\]

would be very cumbersome in terms of derivatives. However, we know that the energies can be approximated by second order Taylor expansion

\[
T(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} \approx \frac{1}{2} \dot{q}_0^T M(q_0) \dot{q}_0,
\]

\[
U(q) = U(q_0) + U'(q_0)(q - q_0) + \frac{1}{2} (q - q_0) U''(q_0)(q - q_0).
\]

We can get these as well by

\[
T(q, \dot{q}) = \frac{1}{2} \sum_{i=1}^{n} m_i (\dot{x}_i^2 + \dot{y}_i^2) + m_i l_i^2 \dot{q}_i^2,
\]

\[
U(q) = g \sum_{i=1}^{n} m_i y_i = g \sum_{i=1}^{n} m_i \sum_{k=1}^{i} l_k \cos q_k,
\]

which in this case will be easier to work with. Note that

\[
U'(q_0) = -g(q_0),
\]

and that \(U''(q_0)\) is a stiffness matrix. Hence, we can obtain the linearized system by

\[
\mathbf{u} = \frac{d}{dt} \frac{\partial (T - U)}{\partial \mathbf{q}} - \frac{\partial (T - U)}{\partial \mathbf{q}},
\]

\[
= M(q_0) \ddot{\mathbf{q}} + U'(q_0) + U''(q_0)(q - q_0)
\]

\[
= M(q_0) \ddot{\mathbf{q}} - g(q_0) + U''(q_0)(q - q_0).
\]
We realize that $U''(q_0)$ is a diagonal matrix with

$$U''_{ii}(q) = -gl_i \cos q_i \sum_{k=i}^n m_k,$$

and, hence, we have a linearized simulator. We can formulate this system as a differential equation of degree 1 with

$$\ddot{p} = \ddot{q}$$

$$\ddot{q} = M_0^{-1} g(q_0) + M_0^{-1} U''_0 q_0 - M_0 U''_0 q + M_0^{-1} u = \dot{p},$$

where $M_0 = M(q_0)$ and $U''_0 = U''(q_0)$. As we have a linear system it holds for piecewise constant actions $u$ that

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \frac{1}{\Delta t} \begin{pmatrix} q_{t+1} - q_t \\ p_{t+1} - p_t \end{pmatrix}$$

and we can write

$$\begin{pmatrix} q_{t+1} \\ p_{t+1} \end{pmatrix} = \left( -\Delta t \cdot M_0^{-1} U''_0 \begin{pmatrix} \Delta t \cdot I \\ I \end{pmatrix} + \Delta t \cdot \begin{pmatrix} 0 \\ M_0^{-1} \end{pmatrix} \right) u$$

$$+ \Delta t \cdot \begin{pmatrix} 0 \\ M_0^{-1} g(q_0) + M_0^{-1} U''_0 q_0 \end{pmatrix} \text{ (Simulator Equation)}$$

4 Linearization around the Balance Point

The balance point, i.e. all joints are directly straight and vertically aligned, is described by $q_0 = 0$. We see that $g(q_0) = 0$ and so the constant term in the Simulator Equation vanishes. The computation of the Mass matrix simplifies to

$$M_{ih}(q_0) = l_h l_i \sum_{k=\max(i,h)}^n m_k,$$

and

$$U''_{ii}(q) = -gl_i \sum_{k=i}^n m_k.$$